

CODES ON LINEAR SECTIONS OF GRASSMANNIANS

JESÚS CARRILLO-PACHECO AND FELIPE ZALDIVAR

ABSTRACT. We study algebraic geometry linear codes defined by linear sections of the Grassmannian variety as codes associated to $\text{FFN}(1, q)$ -projective varieties. As a consequence, we show that Schubert, Lagrangian-Grassmannian, and isotropic Grassmannian codes are special instances of codes defined by linear sections of the Grassmannian variety.

1. INTRODUCTION

Let \mathbb{F}_q be a finite field with q elements, and denote by $\overline{\mathbb{F}}_q$ an algebraic closure of \mathbb{F}_q . For E a vector space over \mathbb{F}_q of finite dimension k , let $\overline{E} = E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ the corresponding vector space over the algebraically closed field $\overline{\mathbb{F}}_q$. We will be considering algebraic varieties in the projective space $\mathbb{P}(\overline{E}) = \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$. Recall now that a projective variety $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ is defined over the finite field \mathbb{F}_q if its vanishing ideal can be generated by polynomials with coefficients in \mathbb{F}_q . Also, a projective variety $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ is *non-degenerate* if X is not contained in a hyperplane of $\mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$. For a projective variety $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ defined over \mathbb{F}_q , we denote by $X(\mathbb{F}_q)$ its set of \mathbb{F}_q -rational points.

The arithmetic counterpart of these geometric concepts is the notion of *non-degenerate projective system*, that is, set of points $\chi \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q)$ not contained in a hyperplane of $\mathbb{P}^{k-1}(\mathbb{F}_q)$. The question of when a non-degenerate projective variety $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ descends to a non-degenerate projective system $X(\mathbb{F}_q) \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q)$ is captured by the so-called $\text{FFN}(1, q)$ -property [1], [6], that is, projective varieties $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ that satisfy that every homogeneous linear polynomial with coefficients in \mathbb{F}_q that vanishes on its set of \mathbb{F}_q -rational points $X(\mathbb{F}_q)$, also vanishes on the whole $X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$. These varieties are important in coding theory, since by [22] and [23], to their sets of \mathbb{F}_q -rational points $X(\mathbb{F}_q) \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q)$ there is associated a non-degenerate $[n, k]_q$ -linear code $C_{X(\mathbb{F}_q)} \subseteq \mathbb{F}_q^n$ of length $n = |X(\mathbb{F}_q)|$, dimension k , and minimum distance

$$d = d(C_{X(\mathbb{F}_q)}) = n - \max\{|X(\mathbb{F}_q) \cap H| : H \text{ is a hyperplane of } \mathbb{P}^{k-1}(\mathbb{F}_q)\}.$$

Moreover, the higher weights $d_r = d_r(C_{X(\mathbb{F}_q)})$ of $C_{X(\mathbb{F}_q)}$ are given by

$$d_r = n - \max\{|X(\mathbb{F}_q) \cap H| : H \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q) \text{ a codimension } r \text{ projective subspace}\}.$$

There are several families of projective algebraic varieties X , defined over a finite field \mathbb{F}_q such that the set of \mathbb{F}_q -rational points $X(\mathbb{F}_q) \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q)$ is a non-degenerate system, [6]. Here we will be interested on Grassmann varieties and some of their subvarieties. Recall that if E is a vector space of dimension m , defined over

2010 *Mathematics Subject Classification.* Primary: 11T71; Secondary: 94Bxx.

Key words and phrases. Algebraic geometry codes; Grassmann codes; Lagrangian-Grassmannian codes; Schubert codes; Higher weights.

\mathbb{F}_q , and $\overline{E} = E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, the Grassmann variety $G(\ell, \overline{E}) = G(\ell, m)$ is the set of all vector subspaces of \overline{E} of dimension ℓ . The Plücker embedding of $G(\ell, m)$ into the projective space $\mathbb{P}(\wedge^\ell \overline{E}) = \mathbb{P}^{k-1}(\overline{\mathbb{F}_q})$, for $k = \binom{m}{\ell}$ is non-degenerate. Moreover, the set of \mathbb{F}_q -rational points $G(\ell, m)(\mathbb{F}_q)$ of the Grassmannian is a non-degenerate projective system in $\mathbb{P}^{k-1}(\mathbb{F}_q)$, see [2]. Hence, it defines an $[n, k, d]_q$ -linear code, where $n = |G(\ell, m)(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q$ (the Gaussian q -binomial coefficient), $k = \binom{m}{\ell}$ and $d = q^\delta$, for $\delta = \ell(m - \ell)$, see [19] and [20].

The main contribution of this paper is to give a uniform construction of codes associated to linear sections of the Grassmann variety as codes given by $\text{FFN}(1, q)$ -projective varieties. As a by-product, we show that the Schubert codes of Ghorpade and Lachaud [10], Lagrangian-Grassmannian codes of the authors [4], and the isotropic Grassmannian codes of Cardinali and Giuzzi in [3], are instances of codes associated to linear sections of the Grassmannian.

The paper is organized as follows: In Section 2 we establish some general facts on linear sections of Grassmannians, where the main results are Propositions 2.6, 2.7, and 2.10. Section 3 reinterprets and introduces examples of algebraic-geometry codes as codes associated to linear sections of Grassmannians in the light of the results of Section 2. In Section 4 we obtain general bounds for the higher weights of these codes.

2. PRELIMINARIES AND LINEAR SECTIONS OF GRASSMANNIANS

Let $X \subseteq \mathbb{P}(\overline{E})$ be an irreducible projective variety, where E is a vector space of finite dimension over a finite field \mathbb{F}_q . Let $\overline{\mathbb{F}_q}$ be an algebraic closure of \mathbb{F}_q and $\overline{E} = E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Let $\overline{K} = \{h \in \overline{E}^* : h(x) = 0 \text{ for all } x \in X\}$, where \overline{E}^* is the dual space of \overline{E} . Let $L(X) = \bigcap_{h \in \overline{K}} \ker h$ be the linear hull of X , that is, the smallest linear subspace of $\mathbb{P}(\overline{E})$ that contains X . Thus, if $\overline{V} = \{x \in \overline{E} : h(x) = 0 \text{ for all } h \in \overline{K}\}$, then, $L(X) = \mathbb{P}(\overline{V})$. For the ring extension $\mathbb{F}_q[x_0, \dots, x_N] \subseteq \overline{\mathbb{F}_q}[x_0, \dots, x_N]$, if $I \subseteq \mathbb{F}_q[x_0, \dots, x_N]$ is an ideal, we denote by $I_{\overline{\mathbb{F}_q}}$ its extension to $\overline{\mathbb{F}_q}[x_0, \dots, x_N]$.

Lemma 2.1. *Let $X \subseteq \mathbb{P}(\overline{E})$ be an irreducible projective variety.*

- (1) *The embedding $X \hookrightarrow L(X)$ is non degenerate.*
- (2) *Let $I_{\overline{\mathbb{F}_q}}(X) = \langle f_1, \dots, f_M, g_1, \dots, g_N \rangle$ be the vanishing ideal of X , where f_i and g_j are homogeneous, $\deg f_i \geq 2$ and $\deg g_j = 1$. Then, $L(X) = Z_{\overline{\mathbb{F}_q}}\langle g_1, \dots, g_N \rangle$.*

Proof. We just need to prove the second part. Clearly, $L(X) \subseteq Z_{\overline{\mathbb{F}_q}}\langle g_1, \dots, g_N \rangle$. Now, if $h \in \overline{K}$, then $h \in I_{\overline{\mathbb{F}_q}}(X)$ and $h = \sum_{i=1}^M \alpha_i f_i + \sum_{j=1}^N \beta_j g_j$, with α_i and β_j polynomials with coefficients in $\overline{\mathbb{F}_q}$. Clearly $h = \sum_{j=1}^N \beta'_j g_j$, where $\beta'_j \in \overline{\mathbb{F}_q}$. Thus, if $x \in Z_{\overline{\mathbb{F}_q}}\langle g_1, \dots, g_N \rangle$, then $h(x) = \sum_{j=1}^N \beta'_j g_j(x) = 0$ and hence $Z_{\overline{\mathbb{F}_q}}\langle g_1, \dots, g_N \rangle \subseteq L(X)$. \square

For the finite dimensional \mathbb{F}_q -vector space E , let $\chi = \{P_1, \dots, P_r\}$ be a finite subset of points of the projective space $\mathbb{P}(E)$. Let $K = \{h \in E^* : h(P_1) = \dots = h(P_r) = 0\}$, where E^* is the dual space of E and the linear forms $h \in E^*$ have coefficients in \mathbb{F}_q . Let $W = \{x \in E : h(x) = 0 \text{ for all } h \in K\}$. Notice that in the last part of the proof of Lemma 2.1 we must have that $\mathbb{P}(W) = Z_{\overline{\mathbb{F}_q}}\langle h_1, \dots, h_s \rangle$, if h_1, \dots, h_s is a basis of K . We quote the following lemma and its corollary from [6]:

Lemma 2.2. ([6, Lemma 2.1]). *With the notation above, $\mathbb{P}(W)$ is the smallest linear subvariety of $\mathbb{P}(E)$ that contains χ .*

Corollary 2.3. ([6, Corollary 1]). *With the notation above, χ is a non-degenerate projective system in $\mathbb{P}(W)$.*

The following consequence is immediate (see also [6, Corollary 3]):

Corollary 2.4. *With the notation above, let B be the matrix of the system of linear equations $h_1 = 0, \dots, h_s = 0$, and C_χ the linear code associated to the nondegenerate projective system χ . Then, C_χ is an $[n, k]_q$ -linear code, where $n = |\chi|$ and $k = s - \text{rank } B$.*

Write $\mathbb{P}(\overline{\mathbb{E}}) = \mathbb{P}^n(\overline{\mathbb{F}}_q)$ with homogeneous coordinates x_0, \dots, x_n . For an $\text{FFN}(1, q)$ -projective variety $X \subseteq \mathbb{P}(\overline{\mathbb{E}})$ defined over the finite field \mathbb{F}_q write its vanishing ideal as $I_{\overline{\mathbb{F}}_q}(X) = \langle f_1, \dots, f_M, g_1, \dots, g_N \rangle_{\overline{\mathbb{F}}_q}$, with $f_i, g_j \in \mathbb{F}_q[x_0, \dots, x_n]$ homogeneous forms with $\deg f_i \geq 2$ and $\deg g_j = 1$. We keep this notation for the rest of this section.

Lemma 2.5. *Let $X \subseteq \mathbb{P}(\overline{\mathbb{E}})$ be an $\text{FFN}(1, q)$ projective variety defined over the finite field \mathbb{F}_q . Then, $X(\mathbb{F}_q)$ is a non-degenerated projective system in $\mathbb{P}(V)$, where $V = \bigcap \ker(g_i)$, over \mathbb{F}_q .*

Proof. If $H \subseteq \mathbb{P}(V)$ is hyperplane, say $H = Z_{\mathbb{F}_q}(h)$ for h a homogeneous linear with coefficients in \mathbb{F}_q and if $X(\mathbb{F}_q) \subseteq H$, that is $h(X(\mathbb{F}_q)) = 0$, since X satisfies the $\text{FFN}(1, q)$ property, then $h(X) = 0$, and hence $h \in I_{\overline{\mathbb{F}}_q}(X)$ and since it is linear, $h = \sum_{j=1}^N b_j g_j$, with $b_j \in \overline{\mathbb{F}}_q$. Therefore, $\overline{H} := Z_{\overline{\mathbb{F}}_q}(h) = \mathbb{P}(\overline{V})$, and thus $H = \overline{H} \cap \mathbb{P}(V) = \mathbb{P}(V)$. \square

Proposition 2.6. *Let X be a projective variety defined over a finite field \mathbb{F}_q , and let $J := \langle f_i, g_j, x_k^q - x_k : 0 \leq k \leq n \rangle_{\overline{\mathbb{F}}_q}$. If h is a homogeneous linear form with coefficients in \mathbb{F}_q such that $h^q - h \in J$ and $h \in \sqrt{J}$, then, $h^q \in J$.*

Proof. Since $h \in \sqrt{J}$, there exists an $m \in \mathbb{N}$ such that $h^m \in J$. We distinguish two cases. Firstly, if $m \leq q$, then $h^q = h^m h^{q-m} \in J$. Secondly, if $m > q$, write $m = nq + r$ with $n, r \in \mathbb{N}$ and $r < q$. We do induction on n : If $n = 1$, then $m = q + r$ and $h^m - h^{r+1} = h^{q+r} - h^{r+1} = h^r(h^q - h) \in J$, since $h^q - h \in J$. Now, since $h^m \in J$, from $h^m - h^{r+1} \in J$ it follows that $h^{r+1} \in J$, with $r < q$, that is $r + 1 \leq q$, and by the first case it follows that $h^q \in J$. Assume that the result holds up to n , i.e., if $h^m \in J$ and $m = sq + r$, with $1 \leq s \leq n$ and $r < q$, then $h^q \in J$.

Now, if $m = (n + 1)q + r$, with $r < q$, the hypothesis $h^q - h \in J$ implies that $h^m - h^{m-q+1} = h^{m-q}(h^q - h) \in J$, and since $h^m \in J$, it follows that $h^{m-q+1} \in J$. Thus, $h^{nq+(r+1)} = h^{m-q+1} \in J$. Hence, if $r + 1 < q$, then $h^q \in J$ by induction hypothesis. Now, if $r + 1 = q$, since $h^{(n+1)q} = h^{nq+q} = h^{nq+r+1} \in J$, and on the other hand, since $h^q - h \in J$, then $h^{(n+1)q} - h^{nq+1} = h^{nq}(h^q - h) \in J$, it follows that $h^{nq+1} \in J$, and by induction hypothesis $h^q \in J$. \square

Proposition 2.7. *Let X be a projective variety defined over a finite field \mathbb{F}_q such that $X(\mathbb{F}_q) \neq \emptyset$. If h is a linear homogeneous form with coefficients in \mathbb{F}_q which vanishes in $X(\mathbb{F}_q)$, then there exists a linear homogeneous form h' with coefficients in $\overline{\mathbb{F}}_q$ such that $h - h' \in \langle g_1, \dots, g_N \rangle_{\overline{\mathbb{F}}_q}$.*

Proof. Let $C(X) \subseteq \mathbb{A}_{\mathbb{F}_q}^{n+1}$ be the affine cone of $X \subseteq \mathbb{P}_{\mathbb{F}_q}^n$. Thus, its vanishing ideal is $I_{\mathbb{F}_q}(C(X)) = I_{\mathbb{F}_q}(X) = \langle f_i, g_j \rangle \subseteq \mathbb{F}_q[x_0, \dots, x_n]$ with $f_i, g_j \in \mathbb{F}_q[x_0, \dots, x_n]$ as before. Then, the set of \mathbb{F}_q -rational points of $C(X)$ is

$$\begin{aligned} C(X)(\mathbb{F}_q) &= C(X) \cap \mathbb{A}^{n+1}(\mathbb{F}_q) = Z_{\mathbb{F}_q} \langle f_i, g_j \rangle \cap Z_{\mathbb{F}_q} \langle x_k^q - x_k : 0 \leq k \leq n \rangle \\ &= Z_{\mathbb{F}_q}(J), \quad \text{for } J = \langle f_i, g_j, x_k^q - x_k : 0 \leq k \leq n \rangle. \end{aligned}$$

By the Nullstellensatz,

$$I_{\mathbb{F}_q}(C(X)(\mathbb{F}_q)) = I_{\mathbb{F}_q} Z_{\mathbb{F}_q}(J) = \sqrt{J_{\mathbb{F}_q}}.$$

Now, let $h \in I_{\mathbb{F}_q}(C(X)(\mathbb{F}_q))$ be a linear form, say $h = a_0 x_0 + \dots + a_n x_n$, with $a_i \in \mathbb{F}_q$. Then, $h^q = a_0^q x_0^q + \dots + a_n^q x_n^q = a_0 x_0^q + \dots + a_n x_n^q$ and thus

$$(1) \quad h^q - h = \sum_{k=0}^n a_k (x_k^q - x_k) \in \langle f_i, g_j, x_k^q - x_k : 0 \leq k \leq n \rangle_{\mathbb{F}_q}.$$

On the other hand, since $h \in I_{\mathbb{F}_q}(C(X)(\mathbb{F}_q)) = \sqrt{J_{\mathbb{F}_q}}$, there exists an $m > 0$ such that $h^m \in J_{\mathbb{F}_q}$. By Proposition 2.6, $h^p \in J_{\mathbb{F}_q}$. Writting h as a linear combination of the polynomials $f_i, g_j, x_k^q - x_k$, with coefficients $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}_q$, we obtain

$$h = \left(\sum_i \alpha_i f_i + \sum_k \gamma_k x_k^q \right) + \left(\sum_j \beta_j g_j + \sum_k \gamma_k x_k \right),$$

and since h is linear and homogeneous, it follows that $(\sum_i \alpha_i f_i + \sum_k \gamma_k x_k^q) = 0$. Thus, $h = \sum_j \beta_j g_j + \sum_k \gamma_k x_k$ where β_j and $\gamma_k \in \mathbb{F}_q$. Hence, if $h' := \sum_k \gamma_k x_k$, where γ_k are as before, it follows that $h - h' \in \langle g_1, \dots, g_N \rangle_{\mathbb{F}_q}$. \square

Corollary 2.8. *Let $X \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$ be a projective variety defined over a finite field \mathbb{F}_q and let H be a hyperplane in $\mathbb{P}^n(\mathbb{F}_q)$ such that $X(\mathbb{F}_q) \subseteq H$. Then, there exists a hyperplane H' in $\mathbb{P}^n(\overline{\mathbb{F}_q})$ such that $X(\mathbb{F}_q) \subseteq H'$ but X is not contained in H' , and moreover $H = \bigcap_{j=1}^N H_j \cap H'$.*

Corollary 2.9. *Let $X \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$ be a projective variety defined over the finite field \mathbb{F}_q . Then, X satisfies the FFN(1, q)-property if and only if for each $h \in I_{\mathbb{F}_q}(X(\mathbb{F}_q))$ we have that $h' \in \langle g_1, \dots, g_N \rangle_{\mathbb{F}_q}$.*

Proof. Clearly, X satisfies the FFN(1, q)-property if and only if for each linear homogeneous form $h \in I_{\mathbb{F}_q}(X(\mathbb{F}_q))$ we have that $h(X) = 0$, and this is equivalent to $h \in \langle g_1, \dots, g_N \rangle_{\mathbb{F}_q}$, which, by Proposition 2.7, it happens if and only if $h' \in \langle g_1, \dots, g_N \rangle_{\mathbb{F}_q}$. \square

Linear Sections of Grassmannians. For an ideal $J \subseteq \mathbb{F}_q[x_0, \dots, x_m]$ denote by $Z_{\mathbb{F}_q}(J) \subseteq \mathbb{P}^m(\mathbb{F}_q)$ and $Z_{\overline{\mathbb{F}_q}}(J) \subseteq \mathbb{P}^m(\overline{\mathbb{F}_q})$ the zero sets of J in $\mathbb{P}^m(\mathbb{F}_q)$ and $\mathbb{P}^m(\overline{\mathbb{F}_q})$, respectively. In particular, for a linear form $h \in \mathbb{F}_q[x_0, \dots, x_m]$, $H = Z_{\mathbb{F}_q}(h) \subseteq \mathbb{P}^m(\mathbb{F}_q)$ and $\overline{H} = Z_{\overline{\mathbb{F}_q}}(h) \subseteq \mathbb{P}^m(\overline{\mathbb{F}_q})$ denote the corresponding hyperplanes. For an ideal $I \subseteq \mathbb{F}_q[x_0, \dots, x_m]$ we will denote by $I_{\overline{\mathbb{F}_q}} \subseteq \overline{\mathbb{F}_q}[x_0, \dots, x_m]$ the corresponding ideal in $\overline{\mathbb{F}_q}[x_0, \dots, x_m]$.

Let E be a vector space of finite dimension m over a finite field \mathbb{F}_q and let $G(\ell, m) = G(\ell, \overline{E}) \subseteq \mathbb{P}(\wedge^\ell \overline{E})$ be the Grassmannian variety embedded, via the Plücker map, in the projective space $\mathbb{P}^{k-1}(\overline{\mathbb{F}_q}) = \mathbb{P}(\wedge^\ell \overline{E})$, where $k = \binom{m}{\ell}$. If

$X \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$ is an irreducible projective variety defined over the finite field \mathbb{F}_q , we say that X is a *linear section of the Grassmannian variety* $G(\ell, m)$ if $X = G(\ell, m) \cap \mathbb{P}(V)$, where $V \subseteq \wedge^\ell \overline{E}$ is a vector subspace. Using the Plücker coordinates p_α for the Grassmannian $G(\ell, m) \subseteq \mathbb{P}^{k-1}(\overline{\mathbb{F}}_q)$, where the indexes α run in $I(\ell, m) = \{\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}^\ell : 1 \leq \alpha_1 < \dots < \alpha_\ell \leq m\}$, consider also the set $I[\ell, m]$ of non-ordered ℓ -tuples of the set $[m] = \{1, \dots, m\}$. The set $I(\ell, m)$ is given the Bruhat order, and for an ordered ℓ -tuple $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$ its support is the set $\text{supp}(\alpha) = \{\alpha_1, \dots, \alpha_\ell\} \in I[\ell, m]$. Following [11], for $\Lambda \subseteq I[\ell, m]$ we define the linear section

$$E_\Lambda = \{p = (p_\alpha) \in G(\ell, m) : p_\alpha = 0 \text{ for all } \alpha \in \Lambda\} = G(\ell, m) \cap \mathbb{P}(V)$$

where $V = Z\langle x_\alpha : \alpha \in \Lambda \rangle$. Let $Q = I(G(\ell, m))$ denote the vanishing ideal of $G(\ell, m)$. In [11] it is shown that if $\Lambda \subseteq I[\ell, m]$ is a close subset, then the vanishing ideal of E_Λ , $I(E_\Lambda) = Q + \langle x_\alpha : \alpha \in \Lambda \rangle$, the ideal generated by Q and the indeterminates corresponding to Λ , is a radical ideal.

Proposition 2.10. *E_Λ satisfies the FFN(1, q)-property*

Proof. Let $E_\Lambda(\mathbb{F}_q)$ be the set of \mathbb{F}_q -rational points of E_Λ and assume that the linear form $h = \sum_{\alpha \in I[\ell, m]} a_\alpha X_\alpha$ vanishes on $E_\Lambda(\mathbb{F}_q)$. By definition of Λ , if $\alpha \in \Lambda$ then $p_\alpha = 0$, and if $\lambda \in I(\ell, m) - \Lambda$ (that is, $\lambda \in I(\ell, m)$ but $\text{supp}(\lambda) \notin \Lambda$), let $\overline{p}_\lambda = (p_\alpha)_{\alpha \in I[\ell, m]}$ in $G(\ell, m)$ such that

$$p_\alpha = \begin{cases} 1 & \text{if } \text{supp}(\alpha) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $h(\overline{p}_\lambda) = a_\lambda = 0$ and hence $h = \sum_{\alpha \in \Lambda} a_\alpha X_\alpha$, that by definition of Λ , vanishes on E_Λ . \square

Definition 2.11. The \mathbb{F}_q -linear code associated to the non-degenerate projective system $E_\Lambda(\mathbb{F}_q)$ of Proposition 2.10 is denoted by C_{E_Λ} and will be called a *linear section code of the Grassmannian associated to the close set Λ* . Its parameters are:

- (1) $n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q - q^\delta - q^{\delta-1} - \dots - q^{\delta-r+1}$, where $\delta = \ell(m - \ell)$ and $r = |\Lambda| \geq 2$,
- (2) $k = \begin{bmatrix} m \\ \ell \end{bmatrix} - \text{rank } B$, where B is the matrix associated to the homogeneous system of linear equations $\{x_\alpha = 0 : \alpha \in \Lambda\}$,
- (3) $\dim E_\Lambda = \delta - 2$.

3. EXAMPLES OF LINEAR SECTIONS AND SCHUBERT CALCULUS

Example 3.1 (Schubert codes). If $\lambda \in I(\ell, m)$ and $\Lambda = \{\lambda\}$, then E_Λ is isomorphic to Ω_λ , the Schubert variety for a fixed flag and $\lambda \in I(\ell, m)$. It is easy to see that

$$\Omega_\lambda = G(\ell, m) \cap Z(x_\beta; \beta \not\leq \lambda)$$

and hence the length of the associated code C_{Ω_λ} is $|\Omega_\lambda(\mathbb{F}_q)|$ and its dimension is $\dim C_{\Omega_\lambda} = \begin{bmatrix} m \\ \ell \end{bmatrix} - \text{rank } B$, where B is the associated matrix to the system of linear equations $Z\{x_\beta : \beta \not\leq \lambda\}$.

Example 3.2 (Schubert unions linear codes). Let $\Omega_{\alpha_1}, \dots, \Omega_{\alpha_s}$ be Schubert varieties in the Grassmannian $G(\ell, m)$, and let

$$\begin{aligned} S_U &= \bigcup_{i=1}^s \Omega_{\alpha_i} = \bigcup_{i=1}^s (G(\ell, m) \cap Z_{\mathbb{F}_q}(x_\beta : \beta \in I(\ell, m), \beta \not\leq \alpha_i)) \\ &= G(\ell, m) \cap \bigcup_{i=1}^s Z_{\mathbb{F}_q}(x_\beta : \beta \in I(\ell, m), \beta \not\leq \alpha_i). \end{aligned}$$

Hence,

$$S_U(\mathbb{F}_q) = G(\ell, m)(\mathbb{F}_q) \cap \bigcup_{i=1}^s Z_{\mathbb{F}_q}(x_\beta : \beta \in I(\ell, m), \beta \not\leq \alpha_i).$$

In [13], it is proved that:

- (1) The linear hull of S_U is $L(S_U) = \bigcup_{i=1}^s Z_{\mathbb{F}_q}(x_\beta : \beta \in I(\ell, m), \beta \not\leq \alpha_i)$.
- (2) If $G_U = \bigcup_{i=1}^s \{x_\beta : \beta \in I(\ell, m), \beta \not\leq \alpha_i\}$, then $\dim L(U) = |G_U|$.
- (3) The number of \mathbb{F}_q -rational points in S_U is

$$|S_U(\mathbb{F}_q)| = \sum_{(\lambda_1, \dots, \lambda_\ell)} q^{\lambda_1 + \dots + \lambda_\ell - \frac{\ell(\ell+1)}{2}}.$$

From (1) it follows that $S_U(\mathbb{F}_q)$ is a nondegenerate projective system in $L(S_U)(\mathbb{F}_q)$. The corresponding non-degenerate \mathbb{F}_q -linear code, denoted by $C_{S_U(\mathbb{F}_q)}$, has parameters $n = |S_U(\mathbb{F}_q)|$ and dimension $k = |G_U|$. By [6, Corollary 4], its minimum distance satisfies $d \leq q^{\dim S_U}$.

Example 3.3 (Lagrangian-Grassmannian codes). Let E be a symplectic vector space over a field F , with non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$, and even dimension $2n$. A vector subspace $W \subseteq E$ is *isotropic* iff $\langle x, y \rangle = 0$ for all $x, y \in W$. Hence, the dimension of W is $\leq n$. The Lagrangian-Grassmannian variety $L(n, 2n)$ is the set

$$L(n, 2n) = \{W \in G(n, 2n) : W \text{ is isotropic}\}.$$

Sending a basis v_1, \dots, v_n of each $W \in G(n, 2n)$ to the class of $v_1 \wedge \dots \wedge v_n$ in $\mathbb{P}(\wedge^n E)$ we obtain the following description

$$L(n, 2n) = \{v_1 \wedge \dots \wedge v_n \in G(n, 2n) : \langle v_i, v_j \rangle = 0 \text{ for all } i, j\}.$$

The number of \mathbb{F}_q -rational points of $L(n, 2n)$, see [4] for example, is

$$|L(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^n (1 + q^i).$$

Now, for the contraction map $f : \wedge^n E \rightarrow \wedge^{n-2} E$, given by

$$v_1 \wedge \dots \wedge v_n \mapsto \sum_{1 \leq r < s \leq n} \langle v_r, v_s \rangle v_1 \wedge \dots \wedge \widehat{v_r} \wedge \dots \wedge \widehat{v_s} \wedge \dots \wedge v_n$$

where \widehat{v} means that the corresponding term is omitted, denote by $\mathbb{P}(\ker f)$ the projectivization of $\ker f$. Under the Plücker embedding, $\mathbb{P}(\ker f)$ is a closed irreducible subset of $\mathbb{P}(\wedge^n E)$ and $\mathbb{P}(\ker f) = Z\langle g_1, \dots, g_N \rangle$ is the zero set of a family of linear homogeneous polynomials g_1, \dots, g_N , and we may assume that the g_i are a minimal set of generators. In [4, Section 3] these linear forms were given explicitly. By [4, Lemma 1], $L(n, 2n) = G(n, 2n) \cap \mathbb{P}(\ker f)$. The set of rational points $L(n, 2n)(\mathbb{F}_q)$ is a non-degenerate projective system in $\mathbb{P}(\ker f)$. Indeed, we can be more precise

about this result, but first we recall from [4, Section 3] that for $\alpha \in I(n, 2n)$ we denote by $\alpha_{rs} \in I(n-2, 2n)$ the sequence obtained from α by deleting the indexes corresponding to r and s . Then, let $\Pi_{\alpha_{rs}} = \sum_{i=1}^n a_{i, \alpha_{rs}, 2n-i+1} X_{i, \alpha_{rs}, 2n-i+1}$, where

$$a_{i, \alpha_{rs}, 2n-i+1} = \begin{cases} 1 & \text{if } |\text{supp}\{i, \alpha_{rs}, 2n-i+1\}| = n, \\ 0 & \text{otherwise,} \end{cases}$$

and $X_\alpha \in F[X_\alpha]_{\alpha \in I(n, 2n)}$ are the corresponding indeterminates. With this notation, $\mathbb{P}(\ker f) \subseteq \mathbb{P}(\wedge^n E)$ is the zero set of the $\Pi_{\alpha_{rs}}$, for all $\alpha_{rs} \in I(n-2, 2n)$ as before. Let e_1, \dots, e_{2n} be the standard symplectic basis of E , that is $\langle e_i, e_{2n-i+1} \rangle = 1$ for $1 \leq i \leq n$, and zero otherwise. For $\alpha = (\alpha_1, \dots, \alpha_n) \in I(n, 2n)$, the tensors $e_\alpha = e_{\alpha_1} \wedge \dots \wedge e_{\alpha_n} \in \wedge^n E$ form the usual basis of this vector space.

Lemma 3.4. *With the above notation, the only homogeneous linear forms h in the dual space $(\wedge^n E)^*$ that vanish on the set of \mathbb{F}_q -rational points $L(n, 2n)(\mathbb{F}_q)$ are linear combinations of the form*

$$\Pi_{\alpha_{rs}} = \sum_{i=1}^n X_{i, \alpha_{rs}, 2n-i+1}$$

Proof. By induction on n , assume first that $n = 2$. Then,

$$I(2, 4) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

and notice that $\{e_{12}, e_{13}, e_{24}, e_{34}\} \subseteq L(2, 4)(\mathbb{F}_q)$ since they are totally decomposable and isotropic. Now, if $h \in (\wedge^2 E)^*$ vanishes on $L(2, 4)(\mathbb{F}_q)$, then $L(2, 4)(\mathbb{F}_q) \subseteq Z(h, \Pi)$, where $\Pi = X_{14} + X_{23}$. Suppose that $h = A_{12}X_{12} + A_{13}X_{13} + A_{14}X_{14} + A_{23}X_{23} + A_{24}X_{24} + A_{34}X_{34}$. Since $h(L(2, 4)(\mathbb{F}_q)) = 0$, then $h = A_{14}X_{14} + A_{23}X_{23}$, and since $X_{14} + X_{23} = 0$, it follows that $h = (A_{14} - A_{23})X_{14}$. By [5], $w = (1, 0, 1, 0, 1) \in L(2, 4)(\mathbb{F}_q)$ and thus $h(w) = (A_{14} - A_{23})1 = 0$, that is $A_{14} = A_{23} =: A$, and consequently $h = A(X_{14} + X_{23}) = A\Pi$, as required. Our induction hypothesis is: For all $k < n$, every $h \in (\wedge^k E)^*$ such that $L(k, 2k)(\mathbb{F}_q) \subseteq Z(h, \Pi_{\alpha_{rs}} : \alpha_{rs} \in I(k-2, 2k))$, must be of the form $h = \sum A_{\alpha_{rs}} \Pi_{\alpha_{rs}}$, for $\alpha_{rs} \in I(k-2, 2k)$. Assume now that $h \in (\wedge^n E)^*$ and $L(n, 2n)(\mathbb{F}_q) \subseteq Z(h, \Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n))$. If $h \in \langle \Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n) \rangle$, we are done. Otherwise, write $h = \sum_{\alpha \in I(n, 2n)} A_\alpha X_\alpha$, with $A_{\bar{\alpha}} \neq 0$, where $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_n) \in I(n, 2n)$. The usual basis e_α , for $\alpha \in I(n, 2n)$, can be written as

$$\mathcal{B} = \{e_{(\beta, \bar{\alpha}_n)} : \beta \in I(n-1, 2n-2), \bar{\alpha}_n \notin \text{supp}(\beta)\} \cup \{e_\alpha : \alpha \in \Phi\}$$

where $\Phi = \{\beta \in I(n-1, 2n-2) : \bar{\alpha}_n \notin \text{supp}(\beta)\}$. Then,

$$h = \sum_{(\beta, \bar{\alpha}_n)} A_{(\beta, \bar{\alpha}_n)} X_{(\beta, \bar{\alpha}_n)} + \sum_{\alpha \in \Phi} A_\alpha X_\alpha$$

with $\beta \in I(n-2, 2n-2)$ y $\bar{\alpha}_n \notin \text{supp}(\beta)$. Let

$$h' = \sum_{(\beta, \bar{\alpha}_n)} A_{(\beta, \bar{\alpha}_n)} X_{(\beta, \bar{\alpha}_n)} \quad \text{and} \quad h'' = \sum_{\alpha \in \Phi} A_\alpha X_\alpha,$$

where clearly $h', h'' \in (\wedge^n E)^*$ and $h = h' + h''$. Let ℓ be the isotropic line generated by $e_{\bar{\alpha}_n}$ and let $U(\ell) = \{L \in L(n, 2n) : \ell \subseteq L\}$. We may identify $U(\ell)$ with the Lagrangian-Grassmannian $L(n-1, \ell^\perp/\ell) \simeq L(n-1, 2n-2)$. Consider the map

$$-\wedge e_{\bar{\alpha}_n} : \wedge^{n-1} E \rightarrow \wedge^n E$$

giving by wedging with $e_{\bar{\alpha}_n}$ on basis elements and then extending linearly. Composing this map with the contraction $f : \wedge^n E \rightarrow \wedge^{n-2} E$, explicetely we have, for $\sum_{\beta \in (n-1, 2n-2)} p_\beta e_\beta \in \wedge^{n-1} E$,

$$\begin{aligned} \left(\sum_{\beta \in I(n-1, 2n-2)} p_\beta e_\beta \right) \wedge e_{\bar{\alpha}_n} &= \sum_{\beta \in I(n-1, 2n-2)} p_{(\beta, \bar{\alpha}_n)} (e_\beta \wedge e_{\bar{\alpha}_n}) \\ &= \sum_{\beta \in I(n-1, 2n-2)} p_{(\beta, \bar{\alpha}_n)} e_{(\beta, \bar{\alpha}_n)} =: w \in \wedge^n E, \end{aligned}$$

where the $\bar{\alpha}_n$ in the sums satisfy that $\bar{\alpha}_n \notin \text{supp}(\beta)$. Applying f we obtain

$$\begin{aligned} f(w) &= \sum_{\beta \in I(n-1, 2n-2)} p_{(\beta, \bar{\alpha}_n)} f(e_{(\beta, \bar{\alpha}_n)}) \\ &= \sum_{\beta \in I(n-1, 2n-2)} p_{(\beta, \bar{\alpha}_n)} \left(\sum_{1 \leq r < s \leq n} \langle e_{\alpha_r}, e_{\alpha_s} \rangle e_{(\beta, \bar{\alpha}_n)_{rs}} \right) \\ &= \sum_{1 \leq r < s \leq n} \left(\sum_{1 \leq \varphi_1 < \varphi_2 \leq n} p_{(\beta, \bar{\alpha}_n)_{rs} \varphi_1 \varphi_2} \langle e_{\varphi_1}, e_{\varphi_2} \rangle \right) e_{(\beta, \bar{\alpha}_n)_{rs}}. \end{aligned}$$

Now, if $w \in \text{Im}(-\wedge e_{\bar{\alpha}_n}) \cap \ker f$, since the $e_{(\beta, \bar{\alpha}_n)_{rs}}$ are linearly independent, we must have that $\sum_{1 \leq \varphi_1 < \varphi_2 \leq n} p_{((\beta, \bar{\alpha}_n)_{rs} \varphi_1 \varphi_2)} \langle e_{\varphi_1}, e_{\varphi_2} \rangle = 0$, where $\langle e_{\varphi_1}, e_{\varphi_2} \rangle = 1$ if and only if $\varphi_1 + \varphi_2 = 2n + 1$, that is if $\varphi_1 = i$, then $\varphi_2 = 2n - i + 1$. Hence, the previous sum is $\sum_{i=1}^n p_{(i, (\beta, \bar{\alpha}_n)_{rs}, 2n-i+1)} = 0$ for all $\beta \in I(n-1, 2n-2)$, again for $\bar{\alpha}_n \notin \text{supp}(\beta)$, up to a permutation on the indexes. Therefore, w satisfies the linear relations $\sum_{i=1}^n X_{(i, (\beta, \bar{\alpha}_n)_{rs}, 2n-i+1)} = 0$, for all $\beta \in I(n-1, 2n-2)$. Put $\Pi_{(\beta, \bar{\alpha}_n)_{rs}} := \sum_{i=1}^n X_{(i, (\beta, \bar{\alpha}_n)_{rs}, 2n-i+1)}$. Clearly, $\{\Pi_{(\beta, \bar{\alpha}_n)_{rs}} : 1 \leq r < s \leq n, \beta \in I(n-1, 2n-2)\} \subseteq \{\Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n)\}$,

$$h'(L(n-1, \ell^\perp/\ell)(\mathbb{F}_q)) = h(U(\ell)(\mathbb{F}_q)) = 0,$$

and by the induction hypothesis

$$h' = \sum_{\beta \in I(n-1, 2n-2)} A_{(\beta, \bar{\alpha}_n)} \Pi_{(\beta, \bar{\alpha}_n)} \in \langle \Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n) \rangle,$$

where $\bar{\alpha}_n \notin \text{supp}(\beta)$ and $h' \neq 0$. Hence, applying the same process to h'' we have that $h'' = h'_1 + h'_2$ with $h'_1 \in \langle \Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n) \rangle$. This process must finish in a finite number of steps. \square

Corollary 3.5. *$L(n, 2n)(\mathbb{F}_q)$ is a non-degenerate system in $\mathbb{P}(\ker f)(\mathbb{F}_q)$ and the Lagrangian-Grassmannian $L(n, 2n)$ is an $\text{FFN}(1, q)$ -projective variety in $\mathbb{P}(\ker f)$.*

We denote by $C_{L(n, 2n)}$ the $[n, k]_q$ nondegenerate linear code induced by the projective system $L(n, 2n)(\mathbb{F}_q)$. Here $n = \prod_{i=1}^n (1 + q^i)$, and $k = \binom{2n}{n} - \text{rank } B$, where B is the matrix associated to the homogeneous system of linear equations $\{\Pi_{\alpha_{rs}} : \alpha_{rs} \in I(n-2, 2n)\}$. A detailed description of B and $\text{rank } B$ is in [7, Sections 3 and 4]. For the minimum distance $d = d(L(n, 2n))$ we have the bound $d < q^{\frac{n(n+1)}{2}}$, see [4]. For some low dimension Lagrangian-Grassmannian codes their weight spectra have been completely determined, for example, for the Lagrangian-Grassmannian $C_{L(2, 4)}$ code, by [6] and [5], see also [3], and for the Lagrangian-Grassmannian $C_{L(3, 6)}$ code in [3].

Example 3.6 (Isotropic Grassmannians). The set-up is the same as in Example 3.3, that is, E is a symplectic vector space of dimension $2n$ over a finite field \mathbb{F}_q and $\overline{E} = E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. For any integer $1 \leq \ell \leq n$, let $IG(\ell, 2n) \subseteq G(\ell, 2n)$ be the set of k -dimensional isotropic vector subspaces of \overline{E} . This is projective subvariety of $\mathbb{P}(\wedge^\ell \overline{E})$, by means of the Plücker embedding. The isotropic Grassmannian $IG(\ell, 2n)$ is a section

$$IG(\ell, 2n) = G(\ell, 2n) \cap L,$$

by a linear subspace $L \subseteq \mathbb{P}(\wedge^\ell \overline{E})$ of codimension $\binom{2n}{\ell-2}$. Hence, its dimension is

$$\dim IG(\ell, 2n) = \binom{2n}{\ell} - \binom{2n}{\ell-2}.$$

Also, the cardinality of its set of \mathbb{F}_q -rational points is

$$|IG(\ell, 2n)(\mathbb{F}_q)| = \prod_{i=0}^{\ell-1} \frac{q^{2n-2i} - 1}{q^{i+1} - 1}.$$

By Proposition 2.10, $IG(\ell, 2n)(\mathbb{F}_q)$ is a projective system in $\mathbb{P}(\wedge^\ell E)$. Thus, as in Definition 2.11, it has an associated $[n, k]_q$ -linear code $C_{IG(\ell, 2n)}$ with parameters $n = |IG(\ell, 2n)(\mathbb{F}_q)|$ and $k = \binom{2n}{\ell}$. This family of codes was introduced and studied in [3].

Example 3.7 (Lagrangian-Schubert codes). For the Schubert codes, in [8], [13], [11] and [25] their parameters are obtained by using a flag of vector spaces in the projective space $\mathbb{P}(\wedge^n E)$. Using similar ideas we look at a new code associated to a Schubert variety over a symplectic vector space. Again, let \mathbb{F}_q be a finite field, $\overline{\mathbb{F}_q}$ an algebraic closure, and E an \mathbb{F}_q -symplectic vector space of dimension $2n$. Let $L(n, 2n)$ be the Lagrangian-Grassmannian variety defined in Example 3.3. Fix a flag of isotropic subspaces of E :

$$(*) \quad 0 \subset W_1 \subset W_2 \subset \cdots \subset W_n \subset E$$

such that $\dim W_i = i$ for $1 \leq i \leq n$. One such flag will be called an *isotropic flag* of E . Notice that since W_n is isotropic of dimension n , then $W_n \in L(n, 2n)$. Thus, an isotropic flag of E is just a complete flag of W_n . Observe that each isotropic flag of E can be extended to a complete flag of E by setting $W_{n+i} = W_{n-i}^\perp$, for $1 \leq i \leq n$. Now, for a partition $\lambda = (\lambda_1, \dots, \lambda_n) \in I(n, 2n)$ and an isotropic flag $(*)$ of E , the *Lagrangian-Schubert* variety is the set

$$L(n, 2n)_\lambda := \{W \in L(n, 2n) : \dim(W \cap W_{n+1-\lambda_i}) \geq i, 1 \leq i \leq \ell(\lambda)\}$$

where $\ell(\lambda) = |\{p \in \{1, \dots, n\} : \lambda_p \neq 0\}|$. $L(n, 2n)_\lambda$ is a subvariety of $L(n, 2n)$ of codimension $|\lambda| := \sum_{p=1}^n \lambda_p$ in $\mathbb{P}(\wedge^n E)$. Now, for a partition $\lambda \in I(n, 2n)$ and an isotropic flag $(*)$ of E , consider the subvarieties $L(n, 2n)$ and $\Omega_\lambda(n, 2n)$. Then, $L(n, 2n)_\lambda = L(n, 2n) \cap \Omega_\lambda(n, 2n)$. For the set of \mathbb{F}_q -rational points $L(n, 2n)_\lambda(\mathbb{F}_q)$, by [6, Lemma 2.2], there exists an irreducible projective subvariety $Z \subseteq L(n, 2n)_\lambda$ such that $L(n, 2n)_\lambda(\mathbb{F}_q) = Z(\mathbb{F}_q)$ and Z is an FFN(1, q)-variety. Therefore, by [6, Corollaries 3 and 4], $L(n, 2n)_\lambda(\mathbb{F}_q)$ defines a linear code $C_{L(n, 2n)_\lambda}$ whose parameters are given as follows: let $r = r(\lambda) = |L(n, 2n)_\lambda|$ be the number of rational points of $L(n, 2n)_\lambda$ over the finite field \mathbb{F}_q . Write $L(n, 2n)_\lambda(\mathbb{F}_q) = \{P_1, \dots, P_r\}$ with the P_i representatives of the corresponding points in $\mathbb{P}(\wedge^n E)$, under the Plücker embedding. Let $K = \{h \in (\wedge^n E)^* : h(P_1) = \cdots = h(P_r) = 0\}$ and $V = \{w \in$

$\wedge^n E : h(w) = 0$ for all $h \in K$ as in the beginning of this Section. Then, by [15, Section 3.2], the length of the code $C_{L(n,2n)_\lambda}$ is $|L(n,2n)_\lambda(\mathbb{F}_q)| = \sum_{\beta \not\leq \gamma} q^{\delta_\beta}$, where δ_β is the dimension of the affine space isomorphic to the corresponding Schubert cell. The dimension of $C_{L(n,2n)_\lambda}$ is $k = \binom{2n}{n} - \dim B$, where B is the matrix associated to the system of linear equations $Z(\Pi_{\alpha_{rs}}, X_\beta : \alpha_{rs} \in I(n-2, 2n), \beta \not\leq \gamma)$. The minimum distance d of $C_{L(n,2n)_\lambda}$ satisfies the bound $d \leq q^{\dim L(n,2n)_\lambda}$, by [6, Corollary 4].

Example 3.8 (Lagrangian-Schubert union codes). For $\lambda_i \in I(n, 2n)$, $1 \leq i \leq r$, let $L_U = \bigcup_{i=1}^r L(n, 2n)_{\lambda_i}$ and $S_U = \bigcup_{i=1}^r \Omega_{\lambda_i}$. Let $H_U = I(n, 2n) - G_U$, for G_U as in Example 3.2. Then,

$$\begin{aligned} L_U &= G(n, 2n) \cap Z(\Pi_{\alpha_{rs}}, X_\beta : \alpha_{rs} \in I(n-2, 2n), \beta \in H_U) = S_U \cap \mathbb{P}(\ker f) \\ &= L(n, 2n) \cap Z(X_\beta : \beta \in H_U). \end{aligned}$$

Now, as in Example 3.7, for the set of \mathbb{F}_q -rational points $L_U(\mathbb{F}_q)$, there exists an irreducible FFN(1, q)-subvariety $Y \subseteq L_U$ such that $Y(\mathbb{F}_q) = L_U(\mathbb{F}_q)$ and a corresponding linear code C_{L_U} whose parameters satisfy: $n = |L_U(\mathbb{F}_q)| = \sum_{\beta \in H_U} q^{\delta_\beta}$, $k = \binom{2n}{n} - \dim B$, where B is the matrix associated to the system of linear equations $Z(\Pi_{\alpha_{rs}}, X_\beta : \alpha_{rs} \in I(n-2, 2n), \beta \in H_U)$. The minimum distance d of C_{L_U} satisfies the bound $d \leq q^{\dim L_U}$.

4. HIGHER WEIGHTS OF THE LAGRANGIAN-GRASSMANNIAN CODES

Now, we address the question of finding bounds for the higher weights of the Lagrangian-Grassmannian code and to do this we use what is known, [10], [11] and [16], for the higher weights $d_r(C(n, 2n))$ of the linear code $C(n, 2n)$ associated to the \mathbb{F}_q -rational points $G(n, 2n)(\mathbb{F}_q)$ of the Grassmannian. By [11, Thm. 4],

$$\begin{aligned} d_r(C(n, 2n)) &\geq q^\delta + \dots + q^{\delta-r+1}, \text{ where } \delta = n(2n-n) = n^2, \\ d_r(C(n, 2n)) &= q^\delta + \dots + q^{\delta-r+1}, \text{ if } 1 \leq r \leq \max\{n, 2n-n+1\} = n+1. \end{aligned}$$

Now for the higher weights $d_r(C_{L(n,2n)})$ of the Lagrangian-Grassmannian code, with the notation of Example 3.3, suppose H is a codimension r linear subvariety of $\mathbb{P}(V)$. Then, H has codimension r' in $\mathbb{P}(\wedge^n E)$, for $r' > r$. Observe now that

$$|L(n, 2n)(\mathbb{F}_q) \cap H| = |G(n, 2n) \cap \mathbb{P}(V) \cap H| = |G(n, 2n)(\mathbb{F}_q) \cap H|$$

and thus, for $H \subseteq \mathbb{P}(V)$:

$$\begin{aligned} \max_{\text{codim } H=r} \{|L(n, 2n)(\mathbb{F}_q) \cap H|\} &= \max_{H \subseteq \mathbb{P}(V)} \{|G(n, 2n)(\mathbb{F}_q) \cap H| : \text{codim } H = r\} \\ &\leq \max_{H \subseteq \mathbb{P}(\wedge^n E)} \{|G(n, 2n)(\mathbb{F}_q) \cap H| : \text{codim } H = r'\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |L(n, 2n)(\mathbb{F}_q)| - \max_{H \subseteq \mathbb{P}(\wedge^n E)} \{|G(n, 2n)(\mathbb{F}_q) \cap H| : \text{codim } H = r'\} \\ \leq |L(n, 2n)(\mathbb{F}_q)| - \max_{H \subseteq \mathbb{P}(V)} \{|L(n, 2n)(\mathbb{F}_q) \cap H| : \text{codim } H = r\}. \end{aligned}$$

Hence,

$$\begin{aligned} |L(n, 2n)(\mathbb{F}_q)| - |G(n, 2n)(\mathbb{F}_q)| + \left(|G(n, 2n)(\mathbb{F}_q)| - \max_{H \subseteq \mathbb{P}(\wedge^n E)} \{|G(n, 2n)(\mathbb{F}_q) \cap H|\} \right) \\ \leq d_r(C_{L(n,2n)}), \end{aligned}$$

where the maximum is taken over all linear subvarieties $H \subseteq \mathbb{P}(\wedge^n E)$ such that $\text{codim } H = r'$. We have proved:

Proposition 4.1. *With the notation above,*

$$d_r(C_{L(n,2n)}) \geq |(L(n,2n)(\mathbb{F}_q)| - |G(n,2n)(\mathbb{F}_q)| + d_{r'}(C(n,2n))$$

and

$$\begin{aligned} |L(n,2n)(\mathbb{F}_q)| - |G(n,2n)(\mathbb{F}_q)| + d_{r'}(C(n,2n)) &\leq d_r(C_{L(n,2n)}) \\ &\leq |L(n,2n)(\mathbb{F}_q)| - \dim V + r, \end{aligned}$$

where $r' = \binom{2n}{n} - \dim V + r$ and

$$|G(n,2n)(\mathbb{F}_q)| = \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{(q^{2n} - 1)(q^{2n} - q) \cdots (q^{2n} - q^{n-1})}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})}.$$

We just note that the second bound is obtained from the generalized Singleton bound [24].

Now, following [10, Section 5], fix a set $T(n,2n) = \{w_1, \dots, w_t\}$ of representatives in $\wedge^n E$ corresponding to points in $G(n,2n)(\mathbb{F}_q)$. Given a subspace S of $\wedge^n E$, we put $g(S) = |S \cap T(n,2n)|$ and let

$$g_r(n,2n) = \max\{g(S) : S \text{ is a codimension } r \text{ subspace of } \wedge^n E\}.$$

For $f : \wedge^n E \rightarrow \wedge^{n-2} E$ as in Section 2, let $V = \ker f$. Then, $|L(n,2n)(\mathbb{F}_q)| = g(V)$ and it is immediate that $g(V) \leq g_r(n,2n)$ for $r = \binom{2n}{n} - \dim V$. Therefore, from [10, Corollary 17] we obtain

$$\begin{aligned} d_r(C(n,2n)) &= |G(n,2n)(\mathbb{F}_q)| - g_r(n,2n) \\ &\leq |G(n,2n)(\mathbb{F}_q)| - g(V) = |G(n,2n)(\mathbb{F}_q)| - |L(n,2n)(\mathbb{F}_q)|. \end{aligned}$$

We have proved:

Proposition 4.2. *If $1 \leq r \leq \binom{2n}{n}$, then*

$$d_r(C(n,2n)) \leq |G(n,2n)(\mathbb{F}_q)| - |L(n,2n)(\mathbb{F}_q)|.$$

Again, following [10], we address the problem of determining the maximum number of points on linear sections $H \cap L(n,2n)(\mathbb{F}_q)$ of the Lagrangian-Grassmannian, for linear subvarieties $H \subseteq \mathbb{P}(\wedge^n E)$ of codimension r . This problem can be translated to the same problem for the Grassmann variety as follows: Let H be a codimension r linear subvariety of the projective space $\mathbb{P}(V) = Z\langle g_1, \dots, g_N \rangle$. We want to calculate the number of points in the intersection $H \cap L(n,2n)(\mathbb{F}_q)$. Now, since H is a codimension r linear subvariety of $\mathbb{P}(V)$, then $H = Z\langle g_1, \dots, g_N, h_1, \dots, h_t \rangle$ is a linear subvariety of $\mathbb{P}(\wedge^n E)$ of codimension $r' > r$, where $\{h_1, \dots, h_t\}$ is a set of linear homogeneous polynomials. Then, the problem we are addressing could be translated to the problem of finding the number of points of the intersection of Grassmannian $G(n,2n)(\mathbb{F}_q)$ and the codimension r' linear subvariety of $\mathbb{P}(\wedge^n E)$. If the linear subvariety of $\mathbb{P}(V)$ is of the form $H_\Lambda = Z\langle g_1, \dots, g_N, x_\alpha : \text{for all } \alpha \in \Lambda \rangle$, where $\Lambda \subseteq I(n,2n)$ is a *close* family (see [10]) with k elements, we have an upper bound for the number of points in the intersection $H_\Lambda \cap L(n,2n)(\mathbb{F}_q)$. Indeed, if $H'_\Lambda = Z\langle x_\alpha : \text{for all } \alpha \in \Lambda \rangle$ is the corresponding linear subvariety of $\mathbb{P}(\wedge^n E)$, then

$$H_\Lambda \cap L(n,2n)(\mathbb{F}_q) = H'_\Lambda \cap G(n,2n)(\mathbb{F}_q) \cap \mathbb{P}(V) \subseteq H'_\Lambda \cap G(n,2n)(\mathbb{F}_q)$$

and using [10, 3.2], we have

$$|H'_\Lambda \cap G(n, 2n)(\mathbb{F}_q)| = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{n^2} - q^{n^2-1} - \dots - q^{n^2-k+1}.$$

We have proved that:

Proposition 4.3. *If Λ is a close family of $I(n, 2n)$ with k elements, then*

$$|H_\Lambda \cap L(n, 2n)(\mathbb{F}_q)| \leq \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{n^2} - q^{n^2-1} - \dots - q^{n^2-k+1}.$$

REFERENCES

- [1] E. Ballico and A. Cossidente, On the Finite Field Nullstellensatz. *Australasian J. Combinatorics* **21** (2000) 57–60.
- [2] E. Ballico and A. Cossidente, Finite Field Nullstellensatz and Grassmannians. *Australasian J. Combinatorics* **24** (2001) 313–315.
- [3] Cardinali, I. and L. Giuzzi, Minimum Distance of Symplectic Grassmann Codes. *Linear Alg. and its Applic.* **488** (2016) 124–134.
- [4] J. Carrillo-Pacheco and F. Zaldivar, On Lagrangian-Grassmannian Codes, *Designs, Codes and Cryptography* **60** (2011) 291–268.
- [5] J. Carrillo-Pacheco, G. Vega and F. Zaldivar, The Weight Distribution of a Family of Lagrangian-Grassmannian Codes, in *C2SI 2015. El Hajji et al. (Eds.)* Lect. Notes Comp. Sci. **9084** (Springer Verlag, 2015) 240–246.
- [6] J. Carrillo-Pacheco and F. Zaldivar, On Codes over $\text{FFN}(1, q)$ -projective varieties, *Advances in Mathematics of Communications*, **10** (2016) 209–220.
- [7] J. Carrillo-Pacheco, F. Jarquín-Zárate, M. Velasco-Fuentes and F. Zaldivar, An Explicit Description in Terms of Plücker Coordinates of the Lagrangian-Grassmannian. Submitted (2016). Preprint: arXiv:1601.07501.
- [8] H. Chen, On the Minimum Distance of Schubert Codes, *IEEE Trans. Inform. Theory* **46** (2000) 1535–1538.
- [9] W. Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry* (Cambridge University Press, 1997).
- [10] S. R. Ghorpade and G. Lachaud, Higher Weights of Grassmann Codes, in *Coding Theory, Cryptography and Related Areas* (Springer-Verlag, 2000) 122–131,
- [11] S. R. Ghorpade and G. Lachaud, Hyperplane sections of Grassmannians and the number of MDS linear codes, *Finite Fields and their Applications* **7** (2001) 468–506.
- [12] G.-M. Hana, Schubert unions and codes from ℓ -step flag varieties, in *Arithmetic, Geometry, and Coding Theory*, (Luminy, 2005), Séminaires et Congrès **21**, Soc. Math. France, Paris (2009) 43–61.
- [13] J. P. Hansen, T. Johnsen, and K. Ranestad, Schubert unions in Grassmann varieties, *Finite Fields and their Applications* **13** (2007) 738–750.
- [14] J. P. Hansen, T. Johnsen, and K. Ranestad, Grassmann codes and Schubert unions, in *Arithmetic, Geometry, and Coding Theory*, (Luminy, 2005), Séminaires et Congrès **21**, Soc. Math. France, Paris (2009), 103–121.
- [15] T. Ikeda, Schubert Classes in the Equivariant Cohomology of the Lagrangian Grassmannian, *Adv. Math.* **215** (2007) 1–23.
- [16] D. Yu Nogin, Codes associated to Grassmannians, in *Arithmetic, Geometry and Coding Theory* (Luminy 1993), Walter de Gruyter, 1996) 145–154.
- [17] D. Yu Nogin, Spectrum of Codes associated with the Grassmannian $G(3, 6)$, *Problems of Information Transmission*, **33** No. 2 (1997) 114–123.
- [18] F. Rodier, Codes from flag varieties over a finite field, *J. Pure Appl. Algebra*. **178** (2003) 203–214.
- [19] C. T. Ryan, An application of Grassmannian varieties to coding theory, *Congr. Numer.* **57** (1987), 257–271.
- [20] C. T. Ryan, Projective codes based on Grassmann varieties, *Congr. Num.* **57** (1987) 273–279.
- [21] C. T. Ryan and K. M. Ryan, The minimum weight of Grassmannian codes $C(k, n)$, *Disc. Appl. Math.* **28** (1990) 149–156.

- [22] M. A. Tsfasman and S. G. Vladut, Geometric approach to higher weights, *IEEE Trans. Inform. Theory* **41** (1995) 1564–1588.
- [23] M.A. Tsfasman, S. G. Vladut and D. Nogin, *Algebraic Geometric Codes: Basic Notions* (American Mathematical Society, 2007).
- [24] V. K. Wei, Generalized Hamming Weights for Linear Codes, *IEEE Trans. Inform. Theory* **37** (1991) 1412–1418.
- [25] X. Xiang, On the Minimum Distance Conjecture for Schubert Codes, *IEEE Trans. Inform. Theory* **54** (2008) 486–488.

ACADEMIA DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE LA CIUDAD DE MÉXICO, 09790, MÉXICO, D. F., MÉXICO.

E-mail address: `jesus.carrillo@uacm.edu.mx`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA-I, 09340, MÉXICO, D. F., MÉXICO

E-mail address: `fz@xanum.uam.mx`